

value.

## FDA for the 1-D Advection Equation

We now wish to form approximations for the 1-D advection equation:  $\frac{\partial f}{\partial t} + c \frac{\partial f}{\partial x} = 0$ .

This is a very common equation and is a crucial component of the PDE's for geophysical motions, such as those in the atmosphere.

### 1. Euler Scheme

A first guess at how to discretize this equation might be to use a forward-in-time derivative for the time portion and a centered-in-space derivative for the space part. We will denote this as the FIT/CIS scheme, or the Euler scheme. The scheme can be

written as,  $\frac{f_j^{n+1} - f_j^n}{\Delta t} + c \frac{f_{j+1}^n - f_{j-1}^n}{2\Delta x} = 0$ . Note that the specification of the spatial derivative is at the "n" time level. We could have easily just have chosen the "n+1" time level for the spatial derivatives (what might this mean?).

*Talyor series analysis* —

Lets first use Talyor series analyses to show what the order of approximation for this FDA of this PDE. Lets write out all the Talyor series expansions we need:

$$f_j^{n+1} = f_j^n + \Delta t \frac{\partial f}{\partial t} + \frac{\Delta t^2}{2!} \frac{\partial^2 f}{\partial t^2} + \frac{\Delta t^3}{3!} \frac{\partial^3 f}{\partial t^3} + H.O.T.$$

$$f_{j\pm 1}^n = f_j^n \pm \Delta x \frac{\partial f}{\partial x} + \frac{\Delta x^2}{2!} \frac{\partial^2 f}{\partial x^2} \pm \frac{\Delta x^3}{3!} \frac{\partial^3 f}{\partial x^3} + H.O.T.$$

We now plug these Talyor series expansion back into the FDA, and try to place terms on the left-hand-side which are those of the PDE, and the remaining terms on the right-hand-side.

$$\frac{1}{\Delta t} \left( f_j^n + \Delta t \frac{\partial f}{\partial t} + \frac{\Delta t^2}{2!} \frac{\partial^2 f}{\partial t^2} + \frac{\Delta t^3}{3!} \frac{\partial^3 f}{\partial t^3} + H.O.T. - f_j^n \right) + \frac{c}{2\Delta x} \left( \begin{array}{l} f_j^n + \Delta x \frac{\partial f}{\partial x} + \frac{\Delta x^2}{2!} \frac{\partial^2 f}{\partial x^2} + \frac{\Delta x^3}{3!} \frac{\partial^3 f}{\partial x^3} + H.O.T. \\ - f_j^n + \Delta x \frac{\partial f}{\partial x} - \frac{\Delta x^2}{2!} \frac{\partial^2 f}{\partial x^2} + \frac{\Delta x^3}{3!} \frac{\partial^3 f}{\partial x^3} - H.O.T. \end{array} \right) = 0$$

$$\frac{\partial f}{\partial t} + c \frac{\partial f}{\partial x} = - \left( \frac{\Delta t}{2!} \frac{\partial^2 f}{\partial t^2} + H.O.T. \right) - \left( \frac{c\Delta x^2}{3!} \frac{\partial^3 f}{\partial x^3} + H.O.T. \right)$$

This approximation is of order  $\Delta t$  and  $\Delta x^2$ , and one can see that if  $\Delta t$  and  $\Delta x \rightarrow 0$ , then all terms on the RHS go to zero, and we recover the original PDE. So this FDA is consistent

with the original PDE.

What about being stable? We can learn more about this if we use the original PDE to change the leading time truncation error term to a spacial truncation error term. Taking the original PDE and taking a derivative with respect to  $x$  and then again taking the original PDE and taking a derivative with respect to time and substitute using these two we get,

$$\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2}, \text{ which is actually a form of the wave equation.}$$

Substituting this in for the time derivative,

$$\frac{\partial f}{\partial t} + c \frac{\partial f}{\partial x} = - \left( \frac{c^2 \Delta t}{2!} \frac{\partial^2 f}{\partial x^2} + HOT \right) - \left( \frac{c \Delta x^2}{3!} \frac{\partial^3 f}{\partial x^3} + HOT \right).$$

Now this tells us something about the behavior of the of the FDA, because the leading error on the RHS is like a diffusion term, except that the sign is reversed  $\rightarrow$  ANTI-diffusion! So this FDA will generate a solution which grows in time, and therefore is not stable. At what wavelengths will the growth be most rapid? (shortest wavelengths).

In fact, the Euler scheme CANNOT be used to represent the advection equation. While the technique of using Talyor series is useful to show this, a more general technique of showing whether a FDA is stable or not will now be shown.

### Stability Analysis

In order to prove whether or not a FDA is stable, a stability analysis is performed. Sometimes this is called a "von Neuman" analysis. Remember that if the PDE solution is bounded, then the FD solution must be bounded as well. The basic idea is that we compare the amplitude of two successive solutions from the finite difference scheme, and make sure that the ratio is less than or equal to 1. Starting with the Euler scheme, we write the variable  $f_j^n$  as an amplitude times a planer wave,  $f_j^n = A^n e^{ikj\Delta x}$ . The  $A$  represents the amplitude of the wave. To determine the stability, we look at the successive solution amplitudes:

$$\left| \frac{f_j^{n+1}}{f_j^n} \right| = \left| \frac{A^{n+1} e^{ikj\Delta x}}{A^n e^{ikj\Delta x}} \right| = \left| \frac{A^{n+1}}{A^n} \right| = |\lambda|, \text{ where } \lambda \text{ is the eigenvalue for this problem.}$$

Since we are analyzing first-order in time advection equation, there is only one eigenvalue. There are three cases for this analysis:

- $|\lambda| > 1$       Solution grows, unstable
- $|\lambda| = 1$       Solution neutral, stable
- $|\lambda| < 1$       Solution decays, stable

which one of these cases matches the analytical PDE solution?

An important aspect of stability analysis is that if you can find ANY wavelength that is unstable, the the scheme is considered unstable.

We start the stability analysis by substituting into  $f_j^{n+1} = f_j^n - \frac{c\Delta t}{2\Delta x}(f_{j+1}^n - f_{j-1}^n)$  the planer wave representation to get:

$$A^{n+1} e^{ikj\Delta x} = A^n e^{ikj\Delta x} - \frac{c\Delta t}{2\Delta x} A^n (e^{ik(j+1)\Delta x} - e^{ik(j-1)\Delta x})$$

$$A^{n+1} e^{ikj\Delta x} = A^n e^{ikj\Delta x} - \frac{c\Delta t}{2\Delta x} A^n e^{ikj\Delta x} (2i \sin k\Delta x)$$

$$A^{n+1} = A^n - \frac{c\Delta t}{2\Delta x} A^n (2i \sin k\Delta x)$$

$$A^{n+1} = A^n (1 - i\sigma), \quad \sigma = \frac{c\Delta t}{\Delta x} \sin k\Delta x$$

$$\frac{A^{n+1}}{A^n} = (1 - i\sigma) = \lambda$$

here  $\lambda$  is the eigenvalue for this 1x1 matrix. If  $\lambda$  is less than or equal to one, then the solution will not amplify. We need to find the magnitude of  $\lambda$ .

$$|\lambda| = (\text{Re}^2 + \text{Im}^2)^{1/2} = (1^2 + \sigma^2)^{1/2} \geq 1$$

so this result again confirms that the Euler scheme is unstable, as the magnitude of  $\lambda$  is always greater than 1, for some value of  $k\Delta x$ ,  $\Delta t$ , and  $c$ . Note that as  $k \rightarrow 0$  ( $L \rightarrow \infty$ ), then  $\sigma \rightarrow 0$  and  $|\lambda| \rightarrow 1$ . How fast do the shorter wave lengths blow up? If the courant number is 0.5, and  $k\Delta x = \pi/4$  (8 $\Delta x$  wave), then

$$|\lambda| = \left(1^2 + \left(\frac{1}{2}\right)^2 \sin^2\left(\frac{\pi}{4}\right)\right)^{1/2} = 1.06$$

$$|\lambda|^{10} = 1.8, \quad |\lambda|^{20} = 3.24, \quad |\lambda|^{100} = 359, \text{ etc. Growth is exponential!}$$

## 2. Upstream Scheme

All is not lost, as there are other schemes which are stable. Let us discretize the 1-D advection equation using a forward time scheme and a backward space difference as in,

$$\frac{f_j^{n+1} - f_j^n}{\Delta t} + c \frac{f_j^n - f_{j-1}^n}{\Delta x} = 0$$

This is known as the upstream scheme. Lets find the truncation error first.

*Talyor Series Analysis*

$$f_j^{n+1} = f_j^n + \Delta t \frac{\partial f}{\partial t} + \frac{\Delta t^2}{2!} \frac{\partial^2 f}{\partial t^2} + \frac{\Delta t^3}{3!} \frac{\partial^3 f}{\partial t^3} + H.O.T.$$

$$f_{j-1}^n = f_j^n - \Delta x \frac{\partial f}{\partial x} + \frac{\Delta x^2}{2!} \frac{\partial^2 f}{\partial x^2} - \frac{\Delta x^3}{3!} \frac{\partial^3 f}{\partial x^3} + H.O.T.$$

substituting in,

$$\frac{1}{\Delta t} \left( f_i^n + \Delta t \frac{\partial f}{\partial t} + \frac{\Delta t^2}{2!} \frac{\partial^2 f}{\partial t^2} + \frac{\Delta t^3}{3!} \frac{\partial^3 f}{\partial t^3} + \text{HOT} - f_i^n \right) + \frac{c}{\Delta x} \left( f_i^n - f_i^n + \Delta x \frac{\partial f}{\partial x} - \frac{\Delta x^2}{2!} \frac{\partial^2 f}{\partial x^2} + \frac{\Delta x^3}{3!} \frac{\partial^3 f}{\partial x^3} - \text{HOT} \right) = 0$$

$$\frac{\partial f}{\partial t} + c \frac{\partial f}{\partial x} = - \left( \frac{\Delta t}{2!} \frac{\partial^2 f}{\partial t^2} + \text{HOT} \right) + \left( \frac{c \Delta x}{2!} \frac{\partial^2 f}{\partial x^2} + \text{HOT} \right).$$

The upstream scheme is of order  $\Delta t$  and  $\Delta x$ , which is a first order scheme in time and space. Again, lets eliminate the time derivative term using the original PDE. Substituting  $\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2}$  in for the time derivative,

$$\frac{\partial f}{\partial t} + c \frac{\partial f}{\partial x} = - \left( \frac{c^2 \Delta t}{2!} \frac{\partial^2 f}{\partial x^2} + \text{HOT} \right) + \left( \frac{c \Delta x}{2!} \frac{\partial^2 f}{\partial x^2} + \text{HOT} \right)$$

$$\frac{\partial f}{\partial t} + c \frac{\partial f}{\partial x} = \frac{c}{2} (\Delta x - c \Delta t) \frac{\partial^2 f}{\partial x^2} + \text{HOT}$$

$$\frac{\partial f}{\partial t} + c \frac{\partial f}{\partial x} = \frac{c_r}{2} (1 - c_r) \frac{\partial^2 f}{\partial x^2} + \text{HOT}$$

The quantity,  $c_r$  is called the Courant number and is a very important non-dimensional number which appears often these analyses. It describes how far a wave will travel in one time step relative to the grid spacing. Most advection schemes (as we shall see) are conditionally stable, that is, they are only stable if the courant number is less or equal to 1.

Now the Talyor series analysis again tells us something about the behavior of the of the FDA. The leading error on the RHS is a diffusion term. So this FDA will generate a solution which has some diffusion built in. At what wavelengths will the decay be most rapid? (shortest wavelengths). Also, what happens if  $c_r > 1$ ? Then the term will have a negative sign, and growth will occur. So right now we know that  $c_r \leq 1$  in order to keep the scheme stable.

### Stability Analysis

Doing the stability analysis:

$$A^{n+1} e^{ikj\Delta x} = A^n e^{ikj\Delta x} - c_r A^n (e^{ikj\Delta x} - e^{ik(j-1)\Delta x})$$

$$\frac{A^{n+1}}{A^n} = 1 - c_r (1 - e^{-ik\Delta x}) = \lambda$$

$$\lambda = 1 - c_r (1 - \cos k\Delta x + i \sin k\Delta x)$$

Finding the magnitude of the eigenvalue,

$$|\lambda| = \left( (1 - c_r + c_r \cos k\Delta x)^2 + c_r^2 \sin^2 k\Delta x \right)^{1/2}$$

$$|\lambda| = (1 - 2c_r(1 - \cos k\Delta x)(1 - c_r))^{1/2}$$

Using this expression, we check conditions which keep  $|\lambda| \leq 1$ . Run through the cases:

- (a) if  $c_r < 0$ , then  $|\lambda| > 1$ . The "downwind" scheme is absolutely unstable.
- (b) if  $c_r > 1$ , then  $|\lambda| > 1$  if  $(1 - \cos k\Delta x) > 0$ , which is always the case. Absolutely unstable again.
- (c) if  $c_r \leq 1$ , then need to check the cosine term. Run through the  $k\Delta x$  possibilities.

if $k\Delta x = 0$ ,	$1 - \cos k\Delta x = 0$ ,	then $ \lambda  = 1$
if $k\Delta x = \pi/2$ ,	$1 - \cos k\Delta x = 1$ ,	then $ \lambda  = (1 - 2c_r(1 - c_r))^{1/2}$
if $k\Delta x = \pi$ ,	$1 - \cos k\Delta x = 2$ ,	then $ \lambda  = (1 - 4c_r(1 - c_r))^{1/2}$

As long as  $0 < c_r \leq 1$ , then  $|\lambda| \leq 1$ . This is called conditional stability, or the CFL condition. Almost all schemes have this type of stability restriction.

Note that the amplitude of the scheme for some cases is much less than 1. Just as in the growth case, we can get exponential decay of the solution as well. Let  $k\Delta x = \pi/4$  and  $c_r = 0.5$ . What is the amplitude after 10, 20, and 100 time steps?

$$|\lambda| = \left( 1 - 2 \times \frac{1}{2} (1 - \cos \frac{\pi}{4}) (1 - \frac{1}{2}) \right)^{1/2} = .924$$

$$|\lambda|^{10} = .45$$

$$|\lambda|^{20} = .20$$

$$|\lambda|^{100} \sim 0$$

Therefore for an  $8\Delta x$  wave, no amplitude remain after 100 time steps.

### Phase Errors

We can also use the stability analysis to tell us something about the phase speed of the each wave component. What should the phase speed be (it should be 'c'). Lets first find the phase speed for the analytical solution.

$$f(x, t) = F(x - ct) = F_0 e^{ik(x-ct)} \quad \text{where } f(x, 0) = F_0 e^{ikx} \quad \text{for a single plane wave}$$

Compare the solutions at successive time steps

$$\frac{f(x, t + \Delta t)}{f(x, t)} = \frac{F_0 e^{ik(x-c(t+\Delta t))}}{F_0 e^{ik(x-ct)}} = e^{-ikc\Delta t}$$

Now the change in phase of the wave per time step will be given by  $\theta_a$

$$\theta_a = \tan^{-1}\left(\frac{\text{Im}}{\text{Re}}\right) = \tan^{-1}\left(\frac{\sin kc\Delta t}{\cos kc\Delta t}\right) = -kc\Delta t$$

This is the analytical phase of the solution. Now the phase speed of a wave is given by  $\omega/k$ . In the expression above,

$$kc\Delta t = \frac{2\pi L}{L} \frac{\Delta t}{T} = \omega_a \Delta t$$

$\theta_a$  describes the change of phase per time step. Since  $\omega = kc$ , note that  $\omega/k = c$ , which is exactly the phase speed you should get! Notice that the phase speed  $\omega/k$  is NOT a function of  $k$ , that means that all the waves travel at the same speed. This is a non-dispersive solution.

In a similar way, we can find the numerical phase of the upstream scheme, *from the*

$$\theta_n = \tan^{-1}\left(\frac{-c_r \sin k\Delta x}{1 - c_r(1 - \cos k\Delta x)}\right) = \omega_n \Delta t$$

Here,  $\omega_n/k$  IS a function of  $k \rightarrow$  therefore each wavelength travels at a different speed. The solution is dispersive!

Now to examine how the waves move, we take the ratio of the analytical phases to the numerical phases.

$$\frac{\theta_n}{\theta_a} < 1 \quad \text{waves move slower than } c$$

$$\frac{\theta_n}{\theta_a} = 1 \quad \text{waves move slower than } c$$

$$\frac{\theta_n}{\theta_a} > 1 \quad \text{waves move faster than } c$$

obviously, the best ratio is where the numerical phase speeds match the analytical phase speeds. For the upstream scheme, we get

$$\frac{\theta_n}{\theta_a} = \frac{\tan^{-1}\left(\frac{-c_r \sin k\Delta x}{1 - c_r(1 - \cos k\Delta x)}\right)}{-kc\Delta t} \quad \text{which does not tell you much, so use a computer}$$

to plot the ratio. You plot the phase speed as a function of  $c_r$  and  $k\Delta x$ .

### 3. Leap Frog Scheme

Since the Euler scheme is unstable, and the upstream scheme too diffusive, other schemes are used for represent the 1-D advection PDE. One of the most commonly (probably most common) used FDA is called the leap frog scheme. It is written as,

$$\frac{f_j^{n+1} - f_j^{n-1}}{2\Delta t} + c \frac{f_{j+1}^n - f_{j-1}^n}{2\Delta x} = 0$$

Notice that the scheme is centered in time as well as centered in space. This will require 2 initial conditions ( $f_j^n, f_j^{n-1}$ ) to begin the calculation. The leap frog scheme is called a "three time level scheme" because the FDA has three time levels present. The Euler and upstream scheme where two-time-level schemes. The additional time level has some important implications.

### Taylor series analysis

Using

$$f_j^{n\pm 1} = f_j^n \pm \Delta t \frac{\partial f}{\partial t} + \frac{\Delta t^2}{2!} \frac{\partial^2 f}{\partial t^2} \pm \frac{\Delta t^3}{3!} \frac{\partial^3 f}{\partial t^3} + H.O.T.$$

$$f_{j\pm 1}^n = f_j^n \pm \Delta x \frac{\partial f}{\partial x} + \frac{\Delta x^2}{2!} \frac{\partial^2 f}{\partial x^2} \pm \frac{\Delta x^3}{3!} \frac{\partial^3 f}{\partial x^3} + H.O.T.$$

we get

$$\frac{\partial f}{\partial t} + c \frac{\partial f}{\partial x} = - \left( \frac{\Delta t^2}{3!} \frac{\partial^3 f}{\partial t^3} + H.O.T. \right) - \left( \frac{c\Delta x^2}{3!} \frac{\partial^3 f}{\partial x^3} + H.O.T. \right)$$

which means that the scheme is of order  $\Delta x^2$  and  $\Delta t^2$ .

### Stability analysis

In order to deal with the additional time level, we need to use some tricks in order to perform the stability analysis. Let

$$g_j^{n+1} = f_j^n$$

$$f_j^{n+1} = g_j^n - \frac{c\Delta t}{\Delta x} (f_{j+1}^n - f_{j-1}^n) \quad \text{and now substitute in the fourier representations as}$$

before,

$$f_j^n = A^n e^{ikj\Delta x}$$

$$g_j^n = B^n e^{ikj\Delta x}$$

getting

$$B^{n+1} e^{ikj\Delta x} = A^n e^{ikj\Delta x}$$

$$A^{n+1} e^{ikj\Delta x} = B^n e^{ikj\Delta x} + c_r A^n (e^{ik(j+1)\Delta x} - e^{ik(j-1)\Delta x})$$

$$A^{n+1} = B^n - A^n c_r 2i \sin k\Delta x$$

$$B^{n+1} = A^n$$

These last two equations can be written as a matrix,

$$\begin{bmatrix} -2i\sigma & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} A^n \\ B^n \end{bmatrix} = \begin{bmatrix} A^{n+1} \\ B^{n+1} \end{bmatrix} \text{ where } \sigma = c_r \sin k\Delta x$$

and now find the eigenvalues of the system:

$$\det \begin{bmatrix} -2i\sigma - \lambda & 1 \\ 1 & -\lambda \end{bmatrix}$$

$$\lambda^2 + 2i\sigma\lambda - 1 = 0$$

$$\lambda_{\pm} = \frac{-2i\sigma \pm \sqrt{-4\sigma^2 + 4}}{2}$$

$$\lambda_{\pm} = -i\sigma \pm \sqrt{1 - \sigma^2}$$

Okay, now lets go through some cases:

(a)  $\sigma > 1$  case:

$$\lambda_{\pm} = -i\sigma \pm i\sqrt{\sigma^2 - 1}$$

$$|\lambda_{+}| = \left( (-\sigma + \sqrt{\sigma^2 - 1})^2 \right)^{1/2} = (2\sigma^2 - 2\sigma\sqrt{\sigma^2 - 1} - 1)^{1/2}$$

let  $\sigma = 1 + \epsilon$

$$|\lambda_{+}| = (2 + 4\epsilon - (2 + 2\epsilon)(\sqrt{2\epsilon}) - 1)^{1/2} = (1 + 4\epsilon - 2\sqrt{2\epsilon})^{1/2} > 1$$

So for  $\sigma > 1$ , get an unstable scheme. It turns out the  $|\lambda_{-}|$  root is conditionally stable for  $\sigma > 1$ , but since we cannot control what sign we get, the  $\sigma > 1$  is considered unstable.

(b)  $\sigma < 1$  case:

$$|\lambda| = (\text{Re}^2 + \text{Im}^2)^{1/2} = (\sigma^2 + (1 - \sigma^2))^{1/2} = 1$$

so for this case we get a NEUTRAL scheme, i.e.,  $|\lambda| = 1$ ! There is no dissipation in this scheme! Hence, we have a conditional stability criteria of  $\sigma \leq 1$  for the leap frog scheme. Since  $\sigma = c_r \sin k\Delta x$ , then we must use the most restrictive  $\sin k\Delta x$  value, so that the stability condition is  $c_r \leq 1$ .

Note that the leap frog scheme has 2 eigenvalues, because there are three time levels

needed in the FDA. The extra eigenvalue is called the "computational mode". This is an artificial mode which is created because of the choice of schemes.

### *Common Advection Schemes Used in Atmospheric Models*

Upstream Scheme:  $f_j^{n+1} = f_j^n - \frac{c\Delta t}{\Delta x}(f_j^n - f_{j-1}^n); c > 0$

Large implicit damping  
Small phase errors

Leap Frog Scheme:  $f_j^{n+1} = f_j^{n-1} - \frac{c\Delta t}{\Delta x}(f_{j+1}^n - f_{j-1}^n)$

No damping  
Large phase errors - highly dispersive  
computational mode

Crowley/Lax-Wendroff Scheme:  $f_j^{n+1} = f_j^n - \frac{1}{2}\left(\frac{c\Delta t}{\Delta x}\right)(f_{j+1}^n - f_{j-1}^n) + \frac{1}{2}\left(\frac{c\Delta t}{\Delta x}\right)^2(f_{j+1}^n + f_{j-1}^n - 2f_j^n)$

Weak damping  
Moderate phase errors - dispersive

### **FDA for the 1-D Diffusion Equation**

We now wish to form approximations for the 1-D advection equation:  $\frac{\partial f}{\partial t} = K \frac{\partial^2 f}{\partial x^2}$ . This is a very common equation and is a crucial component of the PDE's for geophysical motions, such as those in the atmosphere.

Let try a forward in time, centered in space approximation to this PDE.

$$\frac{f_j^{n+1} - f_j^n}{\Delta t} = K \frac{f_{j+1}^n + f_{j-1}^n - 2f_j^n}{\Delta x^2}$$

again this is an Euler scheme, and lets perform a truncation error analysis using Taylor series:

#### *Truncation Error Analysis*

$$f_j^{n+1} = f_j^n + \Delta t \frac{\partial f}{\partial t} + \frac{\Delta t^2}{2!} \frac{\partial^2 f}{\partial t^2} + \frac{\Delta t^3}{3!} \frac{\partial^3 f}{\partial t^3} + H.O.T.$$

$$f_{j\pm 1}^n = f_j^n \pm \Delta x \frac{\partial f}{\partial x} + \frac{\Delta x^2}{2!} \frac{\partial^2 f}{\partial x^2} \pm \frac{\Delta x^3}{3!} \frac{\partial^3 f}{\partial x^3} + H.O.T.$$

substituting in,