Geometric Explanation...
The geometric explanation of the rotation of coordinate axes (here really their unit vectors)
is one that makes intuitive sense. In Figure 1, unit vectors \( \hat{i} \) and \( \hat{j} \) are shown from the
usual viewpoint with \( \hat{i} \) being parallel to the width of the page and \( \hat{j} \) being parallel to the
height of the page.

The way a cartesian coordinate system is oriented, however, is really completely arbitrary. The only
requirement is that all axes (and corresponding unit vectors) be perpendicular to each other.
So, the additional unit vectors \( \hat{I} \) and \( \hat{J} \) introduced in Figure 2 are really just the unit vectors for a
Cartesian coordinate system... but rotated 45° from what we're used to.

Any vector is a vector, right? Of course. In Figure 2, the unit vectors \( \hat{I} \) and \( \hat{J} \) from the original
non-rotated coordinate system remain along with the unit vectors of the rotated coordinate
system, \( \hat{I} \) and \( \hat{J} \). If you just focus on the figure, though, you'll see that \( \hat{I} \) and \( \hat{J} \) can be thought
of as just two ordinary vectors within the original non-rotated coordinate system. This means
that they can be defined by \( \hat{i} \) and \( \hat{j} \), just like any other vector. Also, since the \( \hat{k} \) vector didn't move at
all (it’s still sticking out of the page toward us), then we can say that \( \hat{k} = \hat{K} \).

In this case, \( \hat{I} \) is a vector comprised of two components \( \frac{\sqrt{2}}{2} \hat{i} \) in the \( \hat{i} \) direction and \( \frac{\sqrt{2}}{2} \hat{j} \) in the \( \hat{j} \) direction.
Also, \( \hat{J} \) is a vector comprised of two components \( -\frac{\sqrt{2}}{2} \hat{i} \) in the \( \hat{i} \) direction and \( \frac{\sqrt{2}}{2} \hat{j} \) in the \( \hat{j} \) direction.
This can be shown mathematically, but were doing intuitive geometry here.

So, \( \hat{I} = \frac{\sqrt{2}}{2} \hat{i} + \frac{\sqrt{2}}{2} \hat{j} + 0 \hat{k} \) and \( \hat{J} = -\frac{\sqrt{2}}{2} \hat{i} + \frac{\sqrt{2}}{2} \hat{j} + 0 \hat{k} \). So, we've defined our new unit vectors for our rotated
coordinate system (\( \hat{I}, \hat{J}, \hat{K} \)) in terms of the unit vectors from the non-rotated system (\( \hat{i}, \hat{j}, \hat{k} \)).

So, then, what if we wanted to define \( \hat{i}, \hat{j}, \) and \( \hat{k} \) in terms of \( \hat{I}, \hat{J}, \) and \( \hat{K} \)? Well, just rotate all the unit
vectors 45° in the opposite direction, or clockwise. Now \( \hat{I} \) and \( \hat{J} \) are in the position that \( \hat{i} \) and \( \hat{j} \) originally
were in figure 1. \( \hat{i} \) and \( \hat{j} \) can now be thought of as ordinary vectors in the coordinate system with unit
vectors \( \hat{I} \) and \( \hat{J} \), and can therefore be defined using those unit vectors. \( \hat{k} \) didn't move, so \( \hat{K} = \hat{k} \)

\[
\hat{i} = \frac{\sqrt{2}}{2} \hat{i} + \frac{\sqrt{2}}{2} \hat{j} + 0 \hat{K} \quad \text{and} \quad \hat{j} = \frac{\sqrt{2}}{2} \hat{i} + \frac{\sqrt{2}}{2} \hat{j} + 0 \hat{K}
\]
Mathematical Explanations.

The mathematical explanation really comes down to the mathematical interpretation of our geographic explanation. In the Geographic Explanation, we defined a coordinate system with unit vectors \( \hat{i}, \hat{j}, \hat{k} \) in terms of \( i, j, k \):

\[
\hat{i} = \frac{\sqrt{2}}{2}i + \frac{\sqrt{2}}{2}j + 0k, \quad \hat{j} = -\frac{\sqrt{2}}{2}i + \frac{\sqrt{2}}{2}j + 0k
\]

What, now, if we want to take the ‘reciprocal’ of this and find \( \hat{i} \) and \( \hat{j} \) in terms of \( \hat{i} \) and \( \hat{j} \)? Well, geometrically, all we’re really doing is projecting \( \hat{i} \) onto \( \hat{i} \) and \( \hat{j} \) and \( \hat{j} \) onto \( \hat{i} \) and \( \hat{j} \)

There is a mathematical formula for this, too:

Projection of vector \( \vec{b} \) onto vector \( \vec{a} = \text{Proj}_\boldsymbol{a}(\vec{b}) = \left( \frac{\vec{a} \cdot \vec{b}}{\| \vec{a} \|^2} \right) \vec{a} = \frac{\vec{a} \cdot \vec{b}}{\| \vec{a} \|^2} \hat{a} \)

So, the projection of vector \( \hat{i} \) onto \( \hat{i} \) = \( \frac{\hat{i} \cdot \hat{i}}{\| \hat{i} \|^2} \hat{i} \). Hey, \( \hat{i} \) is a unit vector, so we know has

a magnitude of 1, so the projection is really \( \frac{\hat{i} \cdot \hat{i}}{\| \hat{i} \|^2} \hat{i} = (\hat{i} \cdot \hat{i})\hat{i} \).

Similarly, \( \hat{i} \) onto \( \hat{j} = (\hat{j} \cdot \hat{i})\hat{i}, \) j onto \( \hat{i} = (\hat{i} \cdot \hat{j})\hat{i}, \) j onto \( \hat{j} = (\hat{j} \cdot \hat{j})\hat{j} \)

(\( \hat{k} = \hat{k} \ldots \) so all we have to do is change \( \hat{k} \) to \( \hat{K} \)).

Okay ... okay ... that’s just a bunch of equations. So, let’s see it applied

\[
\begin{align*}
\hat{i} & \text{ in terms of } \hat{i} \text{ and } \hat{j} = \text{proj } \hat{i} \text{ onto } \hat{i} + \text{proj } \hat{i} \text{ onto } \hat{j} \\
\hat{j} & \text{ in terms of } \hat{i} \text{ and } \hat{j} = \text{proj } \hat{j} \text{ onto } \hat{i} + \text{proj } \hat{j} \text{ onto } \hat{j} \\
\hat{i} & = (\hat{i} \cdot \hat{i})\hat{i} + (\hat{j} \cdot \hat{i})\hat{j} + 0\hat{K} \\
\hat{j} & = (\hat{i} \cdot \hat{j})\hat{i} + (\hat{j} \cdot \hat{j})\hat{j} + 0\hat{K} \\
\hat{k} & = \hat{K}
\end{align*}
\]

\[
\begin{align*}
\hat{i} & = \left( \frac{\sqrt{2}}{2}i + \frac{\sqrt{2}}{2}j \right) \hat{i} + \left( -\frac{\sqrt{2}}{2}i + \frac{\sqrt{2}}{2}j \right) \hat{j} + 0\hat{K} \\
\hat{j} & = \left( \frac{\sqrt{2}}{2}i + \frac{\sqrt{2}}{2}j \right) \hat{i} + \left( -\frac{\sqrt{2}}{2}i + \frac{\sqrt{2}}{2}j \right) \hat{j} + 0\hat{K} \\
\hat{k} & = \hat{K}
\end{align*}
\]

\[
\begin{align*}
\hat{i} & = \frac{\sqrt{2}}{2}i + \frac{\sqrt{2}}{2}j + 0\hat{K} \\
\hat{j} & = \frac{\sqrt{2}}{2}i + \frac{\sqrt{2}}{2}j + 0\hat{K}
\end{align*}
\]

This is exactly what we found doing this geometrically.

Also note that the components of \( \hat{i} \) and \( \hat{j} \) are really just the cosines of the angles between those vectors and \( \hat{i} \) and \( \hat{j} \). In other words, there is an angle of 45° between \( \hat{i} \) and \( \hat{I} \), between \( \hat{j} \) and \( \hat{I} \), and between \( \hat{j} \) and \( \hat{J} \) and the \( \cos(45^\circ) = \frac{\sqrt{2}}{2} \).

There is an angle of 135° between \( \hat{i} \) and \( \hat{J} \) and \( \cos(135^\circ) = -\frac{\sqrt{2}}{2} \). These two values correspond to the components of \( \hat{i} \) and \( \hat{j} \) above.
Let's expand on this directional cosines idea ...

Basically, the components of \( \hat{\mathbf{I}} \) are simply the cosines of the angles between \( \hat{\mathbf{i}}, \hat{\mathbf{j}}, \) and \( \hat{\mathbf{k}} \) respectively. The components of \( \hat{\mathbf{J}} \) are simply the cosines of the angles between \( \hat{\mathbf{j}}, \hat{\mathbf{i}}, \) and \( \hat{\mathbf{k}} \) respectively. The components of \( \hat{\mathbf{K}} \) are simply the cosines of the angles between \( \hat{\mathbf{k}}, \hat{\mathbf{i}}, \) and \( \hat{\mathbf{k}} \) respectively.

So, let's let the letter \( c \) with two subscripts (one for each vector) represent the cosine of the angle between two vectors.

Thus, \( c_{iI} = \) cosine of the angle between \( i \) and \( I \).

\[
c_{ij} = \text{cosine of the angle between } i \text{ and } J \\
c_{ik} = \text{cosine of the angle between } i \text{ and } K.
\]

The cosine of the angle between other vectors is defined similarly.

Since the components of \( \hat{\mathbf{I}} \) are simply the cosines of the angles between \( \hat{\mathbf{i}}, \hat{\mathbf{j}}, \) and \( \hat{\mathbf{k}} \) respectively, we can define \( \hat{\mathbf{I}} \) in the following manner:

\[
\hat{\mathbf{I}} = (\text{cosine angl. between } \hat{i} & \hat{j})\hat{i} + (\text{cosine angl. bet. } \hat{j} & \hat{k})\hat{j} + (\text{cosine angl. bet. } \hat{k} & \hat{i})\hat{k}
\]

\( \hat{\mathbf{J}} \) and \( \hat{\mathbf{K}} \) can be defined similarly.

\[
\begin{align*}
\hat{\mathbf{I}} &= (c_{iI})\hat{i} + (c_{jI})\hat{j} + (c_{kI})\hat{k} \\
\hat{\mathbf{J}} &= (c_{iJ})\hat{i} + (c_{jJ})\hat{j} + (c_{kJ})\hat{k} \\
\hat{\mathbf{K}} &= (c_{iK})\hat{i} + (c_{jK})\hat{j} + (c_{kK})\hat{k}
\end{align*}
\]

Revalation ... and really the entire point of this page ...

The angle between, say, \( \hat{i} \) and \( \hat{J} \) is the same as the angle between \( \hat{j} \) and \( \hat{i} \). Well, duh!

This means the cosine of the angle between \( \hat{i} \) and \( \hat{J} \) is the same as the cosine of the angle between \( \hat{j} \) and \( \hat{i} \). Double duh! Do you want a triple duh? Ok ...

Since the cosine of the angle between \( \hat{i} \) and \( \hat{J} \) is the \( \hat{i} \) component of the vector \( \hat{\mathbf{J}} \), then the cosine of the angle between \( \hat{j} \) and \( \hat{i} \) is the \( \hat{j} \) component of the vector \( \hat{\mathbf{i}} \) in a reciprocal relationship.

\[
\begin{align*}
\hat{i} &= (c_{iI})\hat{i} + (c_{jI})\hat{j} + (c_{kI})\hat{k} \\
\hat{j} &= (c_{iJ})\hat{i} + (c_{jJ})\hat{j} + (c_{kJ})\hat{k} \\
\hat{k} &= (c_{iK})\hat{i} + (c_{jK})\hat{j} + (c_{kK})\hat{k}
\end{align*}
\]

This makes finding the reciprocal relationship rather trivial.
Lost, confused, pulling your hair out? Well, quadruple duh!

Let’s apply what we’ve done to the example we’ve been working with.

\[
\begin{align*}
\hat{i} &= \frac{\sqrt{2}}{2} \hat{i} + \frac{\sqrt{2}}{2} \hat{j} + 0 \hat{k} \\
\hat{j} &= -\frac{\sqrt{2}}{2} \hat{i} + \frac{\sqrt{2}}{2} \hat{j} + 0 \hat{k} \\
\hat{k} &= \hat{k}
\end{align*}
\]

\[
\begin{align*}
\hat{i} &= \frac{\sqrt{2}}{2} \hat{i} + \frac{\sqrt{2}}{2} \hat{j} + 0 \hat{K} \\
\hat{j} &= \frac{\sqrt{2}}{2} \hat{i} + \frac{\sqrt{2}}{2} \hat{j} + 0 \hat{K} \\
\hat{K} &= \hat{k}
\end{align*}
\]

\[\frac{\sqrt{2}}{2}\] is the cosine of the angle between \(\hat{i}\) & \(\hat{i}\), between \(\hat{i}\) & \(\hat{j}\), between \(J\) & \(\hat{j}\)

\[-\frac{\sqrt{2}}{2}\] is the cosine of the angle between \(\hat{j}\) and \(\hat{i}\)

\[0\] is the cosine of the angle between \(\hat{k}\) and \(\hat{K}\)

If you’re intuitive (unlike me) … If you look closely at the reciprocal relationships on this page and the one before it… you’ll see that I’ve wasted an inordinate amount of time on this. Look closely! Look closely and you’ll see …

**THE COLUMNS BECOME THE ROWS AND THE ROWS BECOME THE COLUMNS**

This is probably the easiest way to find reciprocal relationships!

I’ve probably spent too much time working on this document … but I hope it helps!