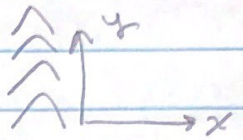
10/1/20
Howie
Bluestein

Use π, θ instead of p, ρ

$$\begin{cases} \pi = \left(\frac{p}{p_0}\right)^{R/c_p} \\ \Rightarrow \frac{1}{\rho}\vec{\nabla} p = c_p \theta \vec{\nabla}\pi \end{cases}$$

$$\begin{cases} \text{no friction or diabatic heating} \\ \frac{\partial}{\partial t} = 0 \quad \text{steady state} \\ \frac{\partial}{\partial y} = 0 \\ R_o >> 1 \Rightarrow \text{neglect } f \end{cases}$$



u-eqn. of motion

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -c_p\theta\frac{\partial \pi}{\partial x}$$

w-eqn of motion

$$u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -c_p\theta\frac{\partial \pi}{\partial z} - g$$

thermo eqn.

$$u\frac{\partial \theta}{\partial x} + w\frac{\partial \theta}{\partial z} = 0$$

incompressible cont. eqn.

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

let

$$\begin{cases} u = \bar{u}(z) + u'(x,z) \\ w = w'(x,z) \qquad (\bar{w} = 0) \\ \theta = \bar{\theta}(z) + \theta'(x,z) \\ \pi = \bar{\pi}(z) + \pi'(x,z) \end{cases}$$

$\bar{\pi}$ ← hydrostatic

now linearize eqns about basic state

(neglect nonlinear terms as small)

So)

(1) $\bar{U}\frac{\partial \bar{U}}{\partial x}^{0} + \bar{U}\frac{\partial U'}{\partial x} + U'\frac{\partial \bar{U}}{\partial x}^{0} + U'\frac{\partial U'}{\partial x}^{\text{small}} = -C_p\bar{\theta}\frac{\partial \bar{\pi}}{\partial x}^{0} - C_p\bar{\theta}\frac{\partial \pi'}{\partial x}$

$+\bar{W}'\underbrace{\frac{\partial \bar{U}}{\partial z}}_{\frac{d\bar{U}}{dz}} + W'\frac{\partial U'}{\partial z}^{\text{small}} \qquad -C_p\theta'\frac{\partial \bar{\pi}}{\partial x}^{0} - C_p\theta'\frac{\partial \pi'}{\partial x}^{\text{small}}$

(2) $\bar{U}\frac{\partial W'}{\partial x} + U'\frac{\partial W'}{\partial x}^{\text{small}} + W'\frac{\partial W'}{\partial z}^{\text{small}} = -C_p\bar{\theta}\frac{\partial \bar{\pi}}{\partial z} - C_p\bar{\theta}\frac{\partial \pi'}{\partial z}$

$-C_p\theta'\frac{\partial \bar{\pi}}{\partial z} - C_p\theta'\frac{\partial \pi'}{\partial z}^{\text{small}} - g$

hydrostatic base state $\Rightarrow$

$-C_p\bar{\theta}\frac{\partial \bar{\pi}}{\partial z} - g = 0 \Rightarrow \frac{\partial \bar{\pi}}{\partial z} = -g/C_p\bar{\theta}$

$= -C_p\bar{\theta}\frac{\partial \pi'}{\partial z} - C_p\theta'\left(\frac{-g}{C_p\bar{\theta}}\right)$

$= -C_p\bar{\theta}\frac{\partial \pi'}{\partial z} + \underbrace{g\frac{\theta'}{\bar{\theta}}}_{B}$

(3) $\bar{U}\frac{\partial \bar{\theta}}{\partial x}^{0} + U'\frac{\partial \bar{\theta}}{\partial x}^{0} + U'\frac{\partial \theta'}{\partial x}^{\text{small}} + \bar{U}\frac{\partial \theta'}{\partial x} + W'\frac{\partial \bar{\theta}}{\partial z} + W'\frac{\partial \theta'}{\partial z}^{\text{small}} = 0$

$\times \frac{g}{\bar{\theta}}$

$\bar{U}\frac{g}{\bar{\theta}}\frac{\partial \theta'}{\partial x} + W'\overbrace{\frac{g}{\bar{\theta}}\frac{\partial \bar{\theta}}{\partial z}}^{N^2} = 0$

$\bar{U}\frac{\partial B}{\partial x} + W'N^2 = 0$

④ $\cancel{\frac{\partial \bar{U}}{\partial x}}^{\,0} + \frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0$

Recapping ⇒

$$\begin{cases} \bar{U}\frac{\partial u'}{\partial x} + w'\frac{d\bar{U}}{dz} + c_p\bar{\theta}\frac{\partial \pi'}{\partial x} = 0 \\ \bar{U}\frac{\partial w'}{\partial x} + c_p\bar{\theta}\frac{\partial \pi'}{\partial z} = B \\ \bar{U}\frac{\partial B}{\partial x} + w'N^2 = 0 \\ \frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0 \end{cases}$$

replace $\bar{\theta} \to \theta_0$

$$P = c_p\bar{\theta}\pi = c_p\theta_0\pi$$

So ⇒

$$\begin{cases} \bar{U}\frac{\partial u'}{\partial x} + w'\frac{d\bar{U}}{dz} + \frac{\partial P'}{\partial x} = 0 \\ \bar{U}\frac{\partial w'}{\partial x} + \frac{\partial P'}{\partial z} = B \\ \bar{U}\frac{\partial B}{\partial x} + N^2w' = 0 \\ \frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0 \end{cases}$$

4 indep eqns in 4 unkowns $u', w', B,$ & P'

to solve for w', eliminate p', then B, then v'

elim. p' by forming z vorticity eqn.

$\frac{\partial}{\partial z}$ of u-eqn. of motion

$-\frac{\partial}{\partial x}$ of z-eqn. of motion

$$\frac{d\bar{u}}{dz}\frac{\partial u'}{\partial x} + \bar{u}\frac{\partial^2 u'}{\partial z \partial x} + \frac{\partial w'}{\partial z}\frac{d\bar{u}}{dz} + w'\frac{d^2\bar{u}}{dz^2} + \frac{\partial^2 p'}{\partial z \partial x} = 0$$

$$\cancel{\frac{d\bar{u}}{dx}}^{0}\frac{\partial w'}{\partial x} + \bar{u}\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 p'}{\partial x \partial z} = \frac{\partial B}{\partial x}$$

$$\frac{d\bar{u}}{dz}\frac{\partial u'}{\partial x} + \bar{u}\frac{\partial}{\partial x}\underbrace{\left(\frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial x}\right)}_{\text{vorticity about y axis}} + \frac{\partial w'}{\partial z}\frac{d\bar{u}}{dz} + w'\frac{d^2\bar{u}}{dz^2} = -\frac{\partial B}{\partial x}$$

– thermo. eqn.

$$\frac{w'N^2}{\bar{u}} = -\frac{\partial B}{\partial x}$$

$$\frac{d\bar{u}}{dz}\underbrace{\left(\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z}\right)}_{0 \text{ cont.}} + \bar{u}\frac{\partial}{\partial x}\left(\frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial x}\right) + w'\frac{d^2\bar{u}}{dz^2} - \frac{N^2 w'}{\bar{u}} = 0$$

$$\bar{u}\frac{\partial}{\partial z}\left(\frac{\partial u'}{\partial x}\right) - \bar{u}\frac{\partial}{\partial x}\left(\frac{\partial w'}{\partial x}\right)$$

$$\bar{u}\frac{\partial}{\partial z}\left(-\frac{\partial w'}{\partial z}\right) - \bar{u}\frac{\partial^2 w'}{\partial x^2}$$

$$\Rightarrow \bar{u}\frac{\partial^2 w'}{\partial z^2} + \bar{u}\frac{\partial^2 w'}{\partial x^2} + \left(\frac{N^2}{\bar{u}} - \frac{d^2\bar{u}}{dz^2}\right)w' = 0$$

$$\div \bar{U} \Rightarrow \frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial z^2} + \underbrace{\left(\frac{N^2}{\bar{U}^2} - \frac{1}{\bar{U}} \frac{d^2 \bar{U}}{dz^2} \right)}_{l^2} w' = 0$$

Scorer parameter (l)

let terrain height $h(x) = h_0 \cos kx$ corrugated sfc!

above reference (sea level? plains?)

amplitude of mtns.

wavenumber

$$z = \cancel{\bar{z}}^{\text{set}=0} + z' \leftarrow h$$

$$z = z(x,t)$$

kinematic l.b.c. $w = \frac{Dz}{Dt} = \frac{dz}{dx} \underbrace{\frac{Dx}{Dt}}_{U}$

so, $w = (\bar{U} + u') \frac{dz}{dx} = (\bar{U} + u') \frac{dh}{dx}$

$$\bar{w} = 0 \Rightarrow w' = -(\bar{U} + \cancel{u'}) h_0 k \sin kx$$

small

$$w'(x, z=h) = -\bar{U} h_0 k \sin kx$$